Enrollment No: ______________________ Exam Seat No:______________________

C.U.SHAH UNIVERSITY

Summer Examination-2017

Subject Name: Group Theory

Subject Code: 5SC04GPE1 **Branch: M.Sc.(Mathematics)**

Semester: 4 **Date: 24/04/2017** **Time: 10:30 to 01:30** **Marks: 70**

Instructions:

(1) Use of Programmable calculator and any other electronic instrument is prohibited.
(2) Instructions written on main answer book are strictly to be obeyed.
(3) Draw neat diagrams and figures (if necessary) at right places.
(4) Assume suitable data if needed.

SECTION – I

Q-1 **Attempt the Following questions** **(07)**

- **a.** Define: Group. **(01)**
- **b.** Define: Sub group. **(01)**
- **c.** Define: Cyclic group. **(01)**
- **d.** Define: Normalizer of element of group. **(01)**
- **e.** Define: Left coset. **(01)**
- **f.** Define: Normal sub group. **(01)**
- **g.** Define: Permutation. **(01)**

Q-2 **Attempt all questions** **(14)**

a) Define Centre of group G. Prove that Centre of group G is subgroup of G. **(05)**

b) If $G = a_0, a_1, \dots\dots, a_6$ **(05)**

where $\begin{cases} a_i \ a_j = a_{i+j} & if \ i+j < 7 \\ \quad\quad = a_{i+j-7} & if \ i+j \geq 7 \end{cases}$ then prove that G is a group under multiplication.

c) Prove that any two right cosets of a subgroup are either disjoint or identical. **(04)**

OR

Q-2 **Attempt all questions** **(14)**

a) Let G be a group. Prove that G is abelian group in each of the following case **(06)**

(i) $a^2 = e, \forall a \in G$

(ii) $(ab)^2 = a^2 b^2 \ \forall a, b \in G$

(iii) $(ab)^i = a^i b^i$ for three consecutive integer i.

b) If $a \in G$ is of order n then prove that $a^m = e$ for some integer m if and only if $n \mid m$. **(05)**

c) Prove that every cyclic group is abelian group. **(03)**

Q-3 **Attempt all questions** **(14)**

a) If H and K are finite subgroups of G of orders $O(H)$ and $O(K)$ then prove that **(05)**

$O(HK) = \frac{O(H)\,O(K)}{O(H \cap K)}$.

	b)	State and prove Lagrange's theorem.	**(05)**
	c)	In a finite group G, prove that $o(a) \mid o(G)$ for each $a \in G$.	**(04)**

OR

Q-3	**a)**	If K is a sub group of G and H is a normal subgroup of G then prove that (i) $K \cap H$ is a normal subgroup of K. (ii) KH is a sub group of G.	**(06)**
	b)	If a cyclic subgroup T of G is normal in G then prove that every subgroup of T is normal in G.	**(05)**
	c)	Let H and K be subgroups of a finite group G and $o(H) > \sqrt{o(G)}$, $o(K) > \sqrt{o(G)}$ then prove that $H \cap K \neq \{e\}$.	**(03)**

SECTION – II

Q-4		**Attempt the Following questions**	**(07)**
	a.	Define: Homomorphism	**(01)**
	b.	Define: Automorphism.	**(01)**
	c.	Define: External direct product	**(01)**
	d.	Define: Simple group.	**(01)**
	e.	Define: Solvable group.	**(01)**
	f.	Define: Invariant of group	**(01)**
	g.	Define: p - sylow group.	**(01)**

Q-5		**Attempt all questions**	**(14)**
	a)	Prove that each permutation $f \in S_n$ can be expressed as a composition of disjoint cycles.	**(06)**
	b)	Let G be a group and suppose G is the internal direct product of $N_1, N_2, \ldots, N_n$. Let $T = N_1 \times N_2 \times \ldots \times N_n$. Then prove that G and T are isomorphic.	**(05)**
	c)	For $n \geq 2$, prove that the set A_n of even permutation in S_n is a sub group of order $\frac{n!}{2}$	**(03)**

OR

Q-5	**a)**	State and prove fundamental theorem of homomorphism.	**(06)**
	b)	If G is finite group then prove that $o(G) = \sum_{i=1}^{k} \frac{o(G)}{o[N(a)]}$	**(05)**
	c)	If N is normal subgroup of group G and $a \in G$ is of order n then prove that the order m of Na in G/N is divisor of n.	**(03)**

Q-6		**Attempt all questions**	**(14)**
	a)	If G is a group then prove that $\mathcal{A}(G)$, the set of all automorphism of G is also a group under the composition of functions	**(06)**
	b)	Prove that $I(G) \cong G/Z$ where $I(G)$ is the group of inner automorphism and Z is the centre of G.	**(05)**
	c)	Let T be an automorphism on group G. If $T(H) = \{T(h): h \in H\}$ then prove that $T(H)$ is a subgroup of G.	**(03)**

OR

Q-6	**a)**	State and prove Sylow's theorem	**(07)**
	b)	State and prove Cayley's theorem.	**(07)**